Stopping times

Math 622

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Reading material: Shreve Section 8.2, Ocone's Lecture 4 notes, part 1

1 Motivation

In financial math, very often and quite naturally, we study random decisions, such as when to exercise your right to buy an option (American call option), or when to accept an offer for the house you are selling (imagine you're putting your house on a market and offer comes in for how much the buyer is willing to pay for the house, which is random). These decisions involve a random time (the time you decide to take action). The time is random because obviously it depends on the path of the stock's price, or of the offers, which are random.
However, there is a common important feature in both cases here: *your decision of when to take action cannot depend on future information.* Mathematically, if we denote $\mathcal{F}(t)$ as the stream of information available to you at time t, and the random time when you take action is τ, then we require:

$$\{\tau \leq t\} \in \mathcal{F}(t).$$

The event $\{\tau \leq t\}$ means you have taken action on or before time t. The event being $\in \mathcal{F}(t)$ then means your decision of taking action on or before time t entirely depends on the information up to time t, i.e. it does not involve future information. Such τ is called a stopping time and it is an important concept to study.

2 Some preliminary

2.1 Discrete vs continuous time

We can model time in 2 ways. Discrete: consider time $n = 0, 1, 2, ..., N$ where N is our terminal time. Continuous: consider time $t \in [0, T]$, where T is our terminal time. Stopping times are defined in both contexts. Generally speaking, discrete time is "easier" to analyze (don't take this statement too literally). The models we will study in Chapter 7,8 are in continuous time. Generally, most of the statements about stopping times have similar versions in both discrete and continuous times. But when one works in continuous time, it is good to pay attention because there will be subtleties that are not present in discrete time.

2.2 Filtration, sigma-algebra and the flow of information

We denote $\mathcal{F}(t), t \in [0, T]$ to be the filtration in the time interval $[0, T]$, which represents the information we have available up to time t. We require:
(i) Each $\mathcal{F}(t)$ is a sigma-algebra.
(ii) If $s < t$ then $\mathcal{F}(s) \subseteq \mathcal{F}(t)$.
Condition (i) is about the closure property of $\mathcal{F}(t)$: if $A_i, i = 1, 2, ...$ is a countable sequence of events (meaning the number of events can potentially be infinite) in $\mathcal{F}(t)$, then A_i^c (not A_i), $\cup_{i=1}^{\infty} A_i$ (some of A_i has happened), $\cap_{i=1}^{\infty} A_i$ (all of A_i have happened) are also in $\mathcal{F}(t)$. We also require $\emptyset, \Omega \in \mathcal{F}(t)$.
Condition (ii) is about the flow of information, intuitively at the present time t we must also have knowledge of the information of the past up to time s as well.
Sometimes we have $\mathcal{F}(0) = \{\emptyset, \Omega\}$. This means any event at time 0 is deterministic. In terms of a random process, this means the process starts out at a deterministic point x, instead of having a random initial distribution.
We can also consider $cF(n), n = 0, 1, ..., N$ as the discrete analog of continuous time filtration. The requirements are the same.

2.3 Stopping time definition

Definition 2.1. *Let τ be a random variable taking values in $[0, T]$ (resp. $\{0, 1, ..., N\}$). We say τ is a stopping time with respect to $\mathcal{F}(t)$ (resp. $\mathcal{F}(n)$) if for* *all $t \in [0,T]$ (resp. for all $n = 0,1,...,N$)*

$$\begin{aligned} \{\tau \le t\} &\in \mathcal{F}(t) \\ (\textit{ resp. } \{\tau \le n\} &\in \mathcal{F}(n)). \end{aligned}$$

Remark 2.2. *Note that the notion of a stopping time is tied to a filtration (similar to the notion of a martingale). It could happen that τ is a stopping time w.r.t a filtration $\mathcal{F}(t)$ but not a stopping time w.r.t another, smaller filtration $\mathcal{G}(t) \subseteq \mathcal{F}(t)$.*

2.4 First important difference between discrete and continuous time

Consider the discrete time. Since if τ is a $\mathcal{F}(n)$ stopping time then $\{\tau < n\} = \{\tau \le n-1\} \in \mathcal{F}(n-1) \subseteq \mathcal{F}(n)$, we have

$$\{\tau \ge n\} = \{\tau < n\}^c \in \mathcal{F}(n)$$

Hence

$$\{\tau = n\} = \{\tau \le n\} \cap \{\tau \ge n\} \in \mathcal{F}(n).$$

Conversely if $\{\tau = n\} \in \mathcal{F}(n)$ *for all* n then $\{\tau \le n\} = \cup_{i=0}^{n} \{\tau = i\} \in \mathcal{F}(n)$, for all n as well. So we can use either conditions: $\{\tau = n\} \in \mathcal{F}(n)$ or $\{\tau \le n\} \in \mathcal{F}(n)$ as definition for stopping time in discrete time.

Now consider the continuous time. By the property of stopping time listed below, it is also true that $\{\tau < t\} \in \mathcal{F}(t)$. So $\{\tau \ge t\} = \{\tau < t\}^c \in \mathcal{F}(t)$. Therefore, if τ is a stopping time then

$$\{\tau = t\} = \{\tau \le t\} \cap \{\tau \ge t\} \in \mathcal{F}(t).$$

However, *it is NOT true that if $\{\tau = t\} \in \mathcal{F}(t)$ for all t then $\{\tau \le t\} \in \mathcal{F}(t)$.* The reason is because in continuous time, we need to write

$$\{\tau \le t\} = \cup_{0 \le s \le t} \{\tau = s\},$$

and the RHS involves an *uncountalbe* union of events, which doesn't have to be contained in the sigma algebra. This explains the choice of using $\{\tau \le t\} \in \mathcal{F}(t)$ as the definition for continuous time.

2.5 Some properties of stopping time

Lemma 2.3. *Let τ_1, τ_2 be stopping times w.r.t. $\mathcal{F}(t)$. Then*
(i) $\{\tau_1 < t\} \in \mathcal{F}(t), \forall 0 \le t \le T$;
(ii) $\min(\tau_1, \tau_2)$ *and* $\max(\tau_1, \tau_2)$ *are stopping times w.r.t F(t).*

Property (i) follows from the fact that

$$\{\tau_1 < t\} = \cup_{n=1}^{\infty}\{\tau_1 \le t - \frac{1}{n}\},$$

and $\{\tau_1 \le t - \frac{1}{n}\} \in \mathcal{F}(t - \frac{1}{n}) \subseteq \mathcal{F}(t), \forall n$. Property ii is left as homework exercise.

3 Some important examples

Example 3.1. *Jump time of a Poisson process*
Let $N(t)$ be a Poisson process. Then

$$\tau_k := \inf\{t \ge 0 : N(t) = k\}$$

are stopping times w.r.t. $\mathcal{F}^N(t)$.

Reason: $\{\tau_k \le t\}$ means the k^{th} jump happened at or before t. But that is the same as at time t, $N(t) \ge k$. Thus

$$\{\tau_k \le t\} = \{N(t) \ge k\} \in \mathcal{F}(t).$$

Example 3.2. *First hitting time to a point of Brownian motion*
Let $b > 0$ be fixed. Define

$$T_b := \inf\{t \ge 0 : W(t) = b\}$$

to be the first time $W(t)$ hits the level b. Note also the convention that $\inf \emptyset = \infty$, *that is if $W(t)$ never hits b then we set $T_b = \infty$. Then T_b is a stopping time w.r.t. $\mathcal{F}^W(t)$.*

The reasoning here is more complicated. Note that $\{T_b \le t\}$ means $W(.)$ has hit b at or before time t. But we cannot infer any property of $W(t)$ (say $W(t) \ge b$ based on this information) because W is *not monotone.*
It is better to look at the complement: $\{T_b > t\}$ which means $W(.)$ has NOT hit b at or before t, which since $W(.)$ starts at 0 at time 0 is equivalent to

$W(s) < b, 0 < s < t$, the information of which intuitively belongs to $\mathcal{F}(t)$. But this is not rigorous, since again there are uncountably many points s in $[0, t]$.
To fix this, we note that a continuous function is uniquely determined by its values on the rationals, which is countable. Combine these facts we can write

$$\begin{aligned}
\{T_b > t\} &= \{W(s) < t, 0 \leq s \leq t\} = \cup_{i=1}^{n}\{W(s) \leq t - \frac{1}{n}, 0 \leq s \leq t\} \\
&= \cup_{i=1}^{n}\{W(s) \leq t - \frac{1}{n}, s \in [0,t] \cap \mathbb{Q}\} \\
&= \cup_{i=1}^{n} \cap_{s\in\mathbb{Q}} \{W(s) \leq t - \frac{1}{n}\},
\end{aligned}$$

and it follows that $\{T_b > t\} \in \mathcal{F}(t)$. Note the subtle fact here that we need to transition from $W(s) < t$ to $W(s) \leq t - \frac{1}{n}$ for some n. The reason is this: if $W(s) < t$ for all s rationals, we can only conclude that $W(s) \leq t$ for all s. But $W(s) \leq t$ for all s rational if and only if $W(s) \leq t$ for all s.
We did not use any special property of Brownian motion besides the fact that it has continuous paths. So

Example 3.3. *First hitting time to a point of a continuous process*
Let $b > 0$ be fixed. Let $X(t)$ be a process starting at 0 with continuous paths. Define

$$T_b := \inf\{t \geq 0 : X(t) = b\}$$

to be the first time $X(t)$ hits the level b. Then T_b is a stopping time w.r.t. $\mathcal{F}^X(t)$.

Example 3.4. *Non example: last hitting time*
Let $b > 0$ be fixed. Let $X(t)$ be a process starting at 0 with continuous paths. Define

$$T_b := \sup\{t \geq 0 : X(t) = b\}$$

to be the last time $X(t)$ hits the level b. Then T_b may NOT be a stopping time w.r.t. $\mathcal{F}^X(t)$.

The reason is this: $\{T_b \leq t\}$ means the last time $X(t)$ hits b is at or before time t. But it is impossible to know whether $X(t)$ will hit b again unless we observe the future paths of $X(t)$, which is forbidden for a stopping time definition. There is an exception: if we know that $X(t)$ is monotone, then once it hits b it will not hit b again. But this is probably the only exception.

Example 3.5. *First hitting time to an open set of a continuous process*
Let $b > 0$ be fixed. Let $X(t)$ be a process starting at 0 with continuous paths. Define

$$S_b := \inf\{t \geq 0 : X(t) > b\}$$

to be the first time $X(t)$ hits the open set (b, ∞). Then S_b may NOT be a stopping time w.r.t $\mathcal{F}^X(t)$.

The reason is very subtle here. It is tempting to write

$$\begin{aligned}\{S_b > t\} &= \{X_s \leq b, 0 \leq s \leq t\} = \{X_s \leq b, s \in \mathbb{Q}\} \\ &= \cap_{s\in\mathbb{Q}}\{X_s \leq b\},\end{aligned}$$

therefore $\{S_b > t\} \in \mathcal{F}(t)$ and S_b is a stopping time. What happens is the first equality is incorect, and it is because of the definition of infimum. It could be the case that at time t, $X(t) = b$ and immediately after t, X crosses over b. Then in this case $S_b = t$ and the event we describe is still in the RHS of the above equation. In other words,

$$\{S_b \geq t\} = \{X_s \leq b, 0 \leq s \leq t\}$$

and we don't have the right inequality to work with here. But note the fact that $S)$ is almost a stopping time. We call it an optional time here.

Remark 3.6. *Another useful way to think of the above situation is to imagine 2 possible paths of $X(s)$: one path ω hits b at time t and crosses over. The other ω'* **follows the exact same path up to time** *t, hits b at time t and immediately reflects down, and let's say never comes back to level b. Then $S_b(\omega) = t$ and $S_b(\omega') = \infty$. Since the two paths are* **the same up to time** *t, it is impossible to tell the event $S_b = t$ by observing $\mathcal{F}(t)$. So S_b cannot be a stopping time. This can be used as a useful, albeit non-rigorous criterion to determine whether a random time is a stopping time.*

Remark 3.7. *In the case $X(t)$ is a Brownian motion, it can be shown that*

$$\mathbb{P}(T_b \neq S_b) = 0,$$

see e.g. Karatzas and Shreve's problem 7.19. In other words, S_b is equal to T_b up to sets of measure 0. Therefore, if we include sets of measure 0 in $\mathcal{F}(t)$, for all t, a process called augmentation of filtration, then S_b is a stopping time with respect to the augmented filtration.

4 Stopped processes

Definition 4.1. *Given a stochastic X and a random time T, we define the stopped process X at time T as*

$$\begin{aligned} X(t \wedge T(\omega))(\omega) &:= X(t)(\omega), t \le T(\omega) \\ &:= X(T)(\omega), t \ge T(\omega). \end{aligned}$$

When T is a stopping time and X is a martingale then the stopped process is also a martingale via the following theorem:

Theorem 4.2. *Let $M(t)$ be a martingale w.r.t. $\mathcal{F}(t)$ with càdlàg paths. Let τ be a stopping time w.r.t $\mathcal{F}(t)$. Then $M(t \wedge \tau)$ is also a martingale w.r.t $\mathcal{F}(t)$.*

This theorem has a discrete time analog:

Theorem 4.3. *Let $M(n)$ be a martingale w.r.t $\mathcal{F}(n)$ and τ a $\mathcal{F}(n)$ stopping time. Then $X(t \wedge n)$ is also a martingale w.r.t $\mathcal{F}(n)$.*

In particular, in the continuous time, when M is a stochastic integral against Brownian motion, then the stopped processed $M(t \wedge \tau)$ is also a martingale when τ is a stopping time. But in this case, we also have an interesting representation of the stopped stochastic integral via the following theorem.

Theorem 4.4. *Let $\mathcal{F}(t)$ be a filtration and $W(t)$ a $\mathcal{F}(t)$ Brownian motion. Let α be an adapted process to $\mathcal{F}(t)$ such that $\int_0^t \alpha(s)dW(s)$ is well-defined. Let τ be a $\mathcal{F}(t)$ stopping time. Denote $M(t) := \int_0^t \alpha(s)dW(s)$. Then $M(t \wedge \tau)$ is a $\mathcal{F}(t)$ martingale. Moreover,*

$$M(t \wedge \tau) = \int_0^{t \wedge \tau} \alpha(s)dW(s) = \int_0^t \mathbf{1}_{[0,\tau)}(s)dW(s).$$

5 Strong Markov property of Brownian motion

It is a well-known fact of Brownian motion that it has independent and stationary increment: if $t > s$ then $W(t) - W(s)$ is independent of $\mathcal{F}(s)$ and has distribution $N(0, t-s)$. In particular, this implies that $W(t) - W(s)$ is a Brownian motion independent of $\mathcal{F}(s)$.

What is interesting is if we replace s by a stopping time, all of these results still hold, except for the technical issue of defining what $\mathcal{F}(\tau)$ means. For our purpose, it is enough to think of $\mathcal{F}(\tau)$ as the sigma algebra containing all information before time τ and we have the following:

Theorem 5.1. *Strong Markov property*
Let W be a Brownian motion and $\mathcal{F}(t)$ a filtration for W. Let τ be a $\mathcal{F}(t)$ stopping time. Then $W(\tau+u)-W(\tau), u \geq 0$ is a Brownian motion independent of all the information in the filtration $\mathcal{F}(t)$ before time τ.

This theorem is called the Strong Markov property because it implies that the Markov property of Brownian motion can be applied to a stopping time as well. Indeed, if we accept the fact that $W(\tau) \in \mathcal{F}(\tau)$ then by the Independence Lemma:

$$E[f\big(W(\tau+u)\big)|\mathcal{F}(\tau)] = g(W(\tau)),$$

where

$$g(x) = E[f\big(W(\tau+u) - W(\tau)\big)].$$

6 Generalization of Remark (3.7) to Ito processes

Remark (3.7) can be generalized to general Ito process: process that can be written as a Rieman integral plus an Ito integral. The intuition here is that the Ito integral has path property similar to that of Brownian motion: very irregular. On the other hand, the Rieman integral has a differentiable ("regular") path. So when the process $X(t)$ hits b, the effect of the stochastic integral part would win out and cause the process to enter b as in the presence of only a Brownian motion.

Theorem 6.1. *Let*

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \sigma(s)dW(s),$$

and suppose that $\mathbb{P}(\sigma(t) \neq 0) = 1$ for all t.
Define

$$\begin{aligned} T_b &:= \inf\{t : X(t) = b\} \\ S_b &:= \inf\{t : X(t) > b\}. \end{aligned}$$

Then $\mathbb{P}(T_b = S_b) = 1$.